

Oscillations of neutrino velocity

Branislav Sazdović, Milovan Vasilčić

Institute of Physics, University of Belgrade, P.O.Box 57, 11001 Belgrade, Serbia

E-mail: sazdovic@ipb.ac.rs, mvasilic@ipb.ac.rs

ABSTRACT: In this paper, we consider the problem of quantum measurement of neutrino velocity. We show, that the well known neutrino flavor oscillations are always accompanied by the oscillations of neutrino velocity. In particular, the velocity of a freely moving neutrino is demonstrated to periodically exceed the speed of light. Unfortunately, the superluminal effect turns out to be too small to be experimentally detected. It is also shown that neutrino velocity significantly depends on the energy, size and shape of the neutrino wave packet. Owing to the big experimental error of the recent experiments, these dependences remained unnoticeable. Finally, we have shown that the recent claims that superluminal neutrinos should loose energy during their flight is not true. Instead, our formula suggests the approximate conservation of energy along neutrino trajectory. All these results have been obtained without violation of special theory of relativity.

KEYWORDS: Neutrino Physics, Beyond Standard Model.

Contents

1. Introduction	1
2. Neutrino dynamics	2
3. Velocity oscillations	5
3.1 Example of a simple wave packet	6
3.2 Graphic illustration	7
4. The shape of the wave packet	10
5. Velocity oscillations of tau neutrinos	13
6. Summary and discussion	14

1. Introduction

The existence of neutrino flavor oscillations is considered a well established fact in contemporary physics [1, 2, 3, 4, 5, 6, 7, 8, 9]. It is caused by the fact that the three known neutrino flavors, ν_e , ν_μ and ν_τ , are not the eigenstates of the Hamiltonian. Instead, they are superpositions of the true eigenstates ν_1 , ν_2 and ν_3 , having sharp masses m_1 , m_2 and m_3 , respectively. As a consequence, the flavors ν_e , ν_μ and ν_τ oscillate during the time evolution of a free neutrino.

The purpose of this work is to examine if similar oscillatory character may appear when it comes to neutrino velocity. It is motivated by the observation that flavor oscillations necessarily imply the oscillations of neutrino masses. Then, owing to the momentum conservation, we expect a freely moving neutrino to have oscillating speed.

The idea that neutrino velocity is closely related to the flavor oscillations is not new. It has been explored in refs [10, 11, 12], with the result that neutrinos are superluminal. However, our analysis shows that the main expression of refs [10, 11, 12] represents just a small correction to our result. In particular, we demonstrate that neutrino velocity considerably depends on the size and shape of the neutrino wave packet.

In this paper, we shall work in the approximation of just two flavors: ν_μ and ν_τ . We shall demonstrate that the velocity of the free muon neutrino indeed has oscillating character. In particular, the neutrino velocity periodically exceeds the speed of light. The maximal value of the detected velocity along neutrino trajectory has also been found. Interestingly enough, the probability to detect maximal velocity is pretty small, and in some cases, goes to zero. At the same time, the obtained formulae turn out to be unexpectedly

sensitive to the shape and size of the neutrino wave packet. This makes the comparison with the known experiments very difficult. For one thing, the exact shape of the experimental wave packets is not known. For the other, we are not convinced that all the packets in the ensemble are identical. Nevertheless, we have tested our formula by comparing its predictions with three recent experiments [13, 14, 15]. For that purpose, the numerical values of our free parameters (such as neutrino energy) are chosen from these experiments, and the undetermined free parameters, related to the wave packet shape and size, were chosen by consulting the literature [16]. As a result, a good agreement with related measurements has been achieved. In particular, the derived energy dependence of the neutrino velocity ($v_{eff} - 1 \sim 1/E^4$ or $1/E^6$) has been shown to remain undetectable in the considered experiments. The same holds for the apparent loss of energy during the flight of superluminal neutrinos [17]. Indeed, we have shown that the rate at which superluminal neutrinos lose their energy is linear in time, but the slope of the graph $E(t)$ is extremely small. This way, the loss of neutrino energy becomes unnoticeable in all terrestrial experiments.

In what follows, we shall use the natural units $\hbar = c = 1$.

2. Neutrino dynamics

To simplify the study of neutrino oscillations, in what follows, we shall adopt two useful approximations. The first is that only two flavors, ν_μ and ν_τ , will be considered to span the internal Hilbert space of the neutrino (the remaining internal degrees of freedom will be neglected). The second is that the neutrino moves along the x -axes, and its dependence on y and z is considered irrelevant. In practice, this means that we reduce our task to a one-dimensional problem. With this in mind, the Hilbert space of the neutrino becomes $\mathcal{H} = C^2 \otimes \mathcal{H}_0$, where C^2 is the internal Hilbert space spanned by two orthonormal vectors $|i\rangle$, ($i = 1, 2$), and \mathcal{H}_0 is orbital Hilbert space spanned by the momentum eigenvectors $|p\rangle$, ($-\infty < p < \infty$). The basis vectors $|i\rangle|p\rangle$ are taken to be the eigenvectors of the Hamiltonian of the free neutrino:

$$\hat{H} |i\rangle|p\rangle = E_i |i\rangle|p\rangle, \quad E_i \equiv \sqrt{m_i^2 + p^2}. \quad (2.1)$$

Thus, the states $|i\rangle|p\rangle$ have the sharp values of mass. On the contrary, the muon and tau neutrinos are not the eigenstates of the Hamiltonian. Precisely, the general ν_μ and ν_τ states have the form

$$|\nu_\mu\rangle = \int dp a(p) (\cos \theta |1\rangle|p\rangle - \sin \theta |2\rangle|p\rangle), \quad (2.2)$$

$$|\nu_\tau\rangle = \int dp b(p) (\sin \theta |1\rangle|p\rangle + \cos \theta |2\rangle|p\rangle), \quad (2.3)$$

where θ is the mixing angle determined by $0.92 \lesssim \sin^2 2\theta \lesssim 1$, and

$$\int dp |a(p)|^2 = \int dp |b(p)|^2 = 1$$

ensures the proper normalization.

In what follows, we shall consider the temporal evolution of *initially pure muon neutrino*. Its generic state vector is given by (2.2), and its evolution is determined by the Hamiltonian (2.1). Thus, we obtain

$$|\nu_\mu(t)\rangle = \int dp a(p) \left(\cos \theta e^{-iE_1 t} |1\rangle|p\rangle - \sin \theta e^{-iE_2 t} |2\rangle|p\rangle \right).$$

The probability density to detect the muon neutrino in point x is then given by

$$P_\mu(x, t) = |\langle \nu_\mu(x) | \nu_\mu(t) \rangle|^2,$$

where $|\nu_\mu(x)\rangle \equiv \cos \theta |1\rangle|x\rangle - \sin \theta |2\rangle|x\rangle$ stands for the eigenstate of the position operator for muon neutrino. Direct calculation then yields

$$P_\mu(x, t) = \frac{1}{2\pi} \left| \int dp a(p) e^{ipx} \left(\cos^2 \theta e^{-iE_1 t} + \sin^2 \theta e^{-iE_2 t} \right) \right|^2. \quad (2.4)$$

To solve this integral, we shall use the reasonable assumption that the distribution $a(p)$ is sharply localized around the value $p = p_0$. This way, the neutrino is characterized by an almost sharp value of momentum. Then, we can expand the energy $E_i(p) \equiv \sqrt{m_i^2 + p^2}$ in a power series around $p = p_0$, and keep only linear terms. We shall also introduce the group velocities

$$v_i \equiv \frac{dE_{i0}}{dp_0},$$

and the shorthand notation $E_{i0} \equiv E_i(p_0)$. With these approximations, the integration in (2.4) leads to

$$P_\mu(x, t) = \left| A_1 \cos^2 \theta e^{-i(E_{10} - v_1 p_0)t} + A_2 \sin^2 \theta e^{-i(E_{20} - v_2 p_0)t} \right|^2, \quad (2.5)$$

where $A_i \equiv A(v_i t - x)$, and

$$A(\tau) \equiv \frac{1}{\sqrt{2\pi}} \int dp a(p) e^{-ip\tau}.$$

The amplitude $A(\tau)$ in the coordinate space is the exact Fourier transform of the momentum amplitude $a(p)$. We have already assumed that the momentum of the neutrino wave packet is well localized around $p = p_0$. To take this explicitly into account, we shall take the neutrino wave packet in the form

$$A(\tau) = \rho(\tau) e^{-i\tau p_0}, \quad (2.6)$$

where the modulus $\rho(\tau)$ is localized around $\tau = 0$, and the wavelength $2\pi/p_0$ is much smaller than the packet size. The latter ensures a small uncertainty of the packet momentum. In what follows, we shall see how our results depend on the size and shape of the neutrino wave packet.

The values of energy and velocity of the two terms in (2.5) differ as a consequence of different masses they carry. To estimate their difference, we introduce

$$\omega \equiv \frac{E_{20} - E_{10}}{2}, \quad \Delta v \equiv v_2 - v_1, \quad \Delta m^2 \equiv m_2^2 - m_1^2.$$

In the ultrarelativistic limit $p_0 \gg m_i$, suitable for the description of neutrinos, E_{i0} is further decomposed as

$$E_{i0} = p_0 + \frac{m_i^2}{2p_0} + \dots \Rightarrow \omega = \frac{\Delta m^2}{4p_0} + \dots, \quad (2.7)$$

and the velocities v_i take the form

$$v_i = 1 - \frac{m_i^2}{2p_0^2} + \dots \Rightarrow \Delta v = -\frac{2\omega}{p_0} + \dots. \quad (2.8)$$

Let us now analyze the probability density (2.5) in more detail. First, observe that the velocities v_1 and v_2 , although close to the speed of light, are different from each other. Owing to this, the two initially overlapping packets will gradually separate. After a long enough time, we shall see two distinct neutrino wave packets. To estimate the time needed for the separation of the two packets, we make use of the packet size. Then, the time needed for their minimal separation (when the distance between the packets reaches the packet size) is given by

$$t = \frac{2\ell}{|\Delta v|} = \frac{4\ell p_0^2}{\Delta m^2},$$

where ℓ is half the size of the wave packet. Using the numerical data from the recent experiments [13, 14, 15] (as shown in Table 2 of the last section), we find

$$t \gtrsim 5 \text{ min},$$

telling us that the two wave packets in (2.5) practically coincide in any terrestrial experiment ($5 \text{ min} \approx 10^8 \text{ km}$). For this reason, in what follows, we shall simplify our considerations by adopting the restriction

$$t|\Delta v| \ll \ell.$$

Note that this restriction still allows for very long flights ($x \sim 10^6 \text{ km}$).

Let us evaluate the amplitudes in the first order in the small parameter $t\Delta v$. To this end, we introduce a new time coordinate τ , and the average velocity \bar{v} , defined by

$$t \equiv \frac{x}{\bar{v}} + \tau, \quad \bar{v} \equiv \frac{v_1 + v_2}{2}. \quad (2.9)$$

The new time coordinate measures time relative to the moment a particle with the average velocity \bar{v} arrives at x . Having in mind that $\bar{v} \approx 1$, we can say that τ *measures the neutrino delay as compared to the arrival time of the photon*. The moduli of the amplitudes now become:

$$\begin{aligned} \rho(v_1 t - x) &= \rho(\bar{v}\tau - \frac{\Delta v}{2}t) \approx \rho(\bar{v}\tau) - \frac{t\Delta v}{2}\rho'(\bar{v}\tau), \\ \rho(v_2 t - x) &= \rho(\bar{v}\tau + \frac{\Delta v}{2}t) \approx \rho(\bar{v}\tau) + \frac{t\Delta v}{2}\rho'(\bar{v}\tau). \end{aligned}$$

Owing to the smallness of the factor $|\frac{\Delta v}{2}|$, the second term is expected to be much smaller than the first one. Indeed, the numerical value of this factor in the recent experiments

[13, 14, 15] is less than 10^{-22} . At the same time, the phase of A , as defined in (2.6), is subject to no approximation at all.

With the adopted approximations, the probability density (2.5) takes the form

$$P_\mu(x, t) = \rho^2(\bar{v}\tau) (1 - \sin^2 2\theta \sin^2 \omega t) - [\rho^2(\bar{v}\tau)]' \cos 2\theta \frac{\Delta v}{2} t, \quad (2.10)$$

where $\bar{v}\tau \equiv \bar{v}t - x$, and the prime denotes derivative with respect to the argument. As the realistic wave packets have finite size, it is natural to assume that the amplitude $\rho(\tau)$ is localized in the interval $-\ell < \tau < \ell$. This way, the time coordinate τ is restricted by the packet size, which is typically much smaller than x . In what follows, we shall adopt the reasonable restriction

$$|\tau| < \ell \ll x. \quad (2.11)$$

3. Velocity oscillations

In this section, we shall study the motion of muon neutrino by studying the spacetime dependence of its probability distribution (2.10). To this end, let us first place the neutrino detector in a fixed position x . This way, the probability to detect the muon neutrino becomes a function of time, only. The *moment neutrino arrives at the detector is determined as the time the probability density (2.10) reaches its maximum in the point x* . The needed time of arrival is then obtained by solving the equation

$$\frac{\partial P_\mu}{\partial t} = 0.$$

With the approximation $\bar{v} \approx 1$, it ultimately leads to

$$\frac{\partial_\tau \rho^2}{\rho^2} - \frac{\partial_\tau^2 \rho^2}{\rho^2} \tau_2(t) + \frac{4}{\ell^2} \tau_1(t) = 0, \quad (3.1)$$

with

$$\tau_1(t) \equiv -\frac{\omega \ell^2}{4} \frac{\sin^2 2\theta \sin 2\omega t}{1 - \sin^2 2\theta \sin^2 \omega t + \frac{\omega}{p_0} \cos 2\theta}, \quad (3.2)$$

$$\tau_2(t) \equiv -\frac{\omega}{p_0} \frac{\cos 2\theta}{1 - \sin^2 2\theta \sin^2 \omega t + \frac{\omega}{p_0} \cos 2\theta} t. \quad (3.3)$$

Before we continue, note that the restriction (2.11) allows us to easily switch between the t and x coordinates. Indeed, after taking into account $|\tau| \ll x$, and $\bar{v} \approx 1$, the equation (2.9) shows that

$$t \approx x.$$

With this, every solution of (3.1), which is originally in the form $\tau = \tau(t)$, can approximately be rewritten in the form $\tau = \tau(x)$.

3.1 Example of a simple wave packet

The above analysis is the most we can do without specifying the amplitude A . As it turns out, the solution $\tau = \tau(x)$ depends a great deal on the size and shape of the wave packet. Before we carefully examine this dependence in the next section, let us describe the simple example of a particular wave packet. Up to the normalization constant, we define the modulus of the amplitude (2.6) as

$$\rho(\tau) \propto \begin{cases} 1 - \frac{\tau^2}{\ell^2}, & -\ell < \tau < \ell \\ 0, & \text{otherwise,} \end{cases} \quad (3.4)$$

where the parameter ℓ determines the packet size. To simplify calculations, we shall work in the approximation $\tau \ll \ell$. First, we multiply the equation (3.1) with $\ell^2 - \tau^2$, and expand it in a power series of τ/ℓ . Then, we drop terms proportional to τ^4/ℓ^4 , whereupon the equation (3.1) becomes the quadratic equation

$$(\tau_1 + 2\tau_2)\tau^2 + \ell^2\tau - (\tau_1 + \tau_2)\ell^2 = 0. \quad (3.5)$$

As seen from (3.5), $\tau_i \rightarrow 0$ implies $\tau \rightarrow 0$, which uniquely determines the physical solution of the quadratic equation to be

$$\tau(x) = -\frac{\ell^2}{2(\tau_1 + 2\tau_2)} \left(1 - \sqrt{1 + 4\frac{\tau_1 + \tau_2}{\ell} \cdot \frac{\tau_1 + 2\tau_2}{\ell}} \right). \quad (3.6)$$

On the other hand, if we restrict our considerations to the linear approximation in τ/ℓ , the equation (3.5) yields

$$\tau(x) = \tau_1(x) + \tau_2(x). \quad (3.7)$$

What is immediately seen is that $\tau_1(x)$ is a *periodic function of x* with the period

$$L = \frac{\pi}{\omega}. \quad (3.8)$$

(For illustration, the neutrino energy of 17 GeV yields $L \approx 18,000$ km). The term $\tau_2(x)$ is linear in x , and the coefficient in front of it is periodic with the same period L . In fact, this term is the main result of Refs. [10, 11, 12]. It does not depend either on the size or on the shape of the wave packet. Owing to the linear dependence on x , the term τ_2 can exceed τ_1 in the limit $x \rightarrow \infty$. Let us accurately estimate the conditions needed to prevent τ_2 from dominating τ_1 . Thus, we start with $|\tau_2| \ll |\tau_1|$, and obtain

$$\frac{\pi}{4} - \theta \ll \frac{p_0\ell^2}{8} \frac{|\sin 2\omega x|}{x}. \quad (3.9)$$

Generally, this condition can not be fulfilled for all distances. However, if we avoid distances which are too close to $nL/2$, the requirement (3.9) can be solved for x . For demonstration purposes, let us take data from the recent experiments [13, 14, 15], as shown in Table 2. Then, one finds $p_0\ell^2/8 \gtrsim 1.15 \cdot 10^8$ km, and $\frac{\pi}{4} - \theta \lesssim 0.14$, which, in the worst scenario,

yields $x \ll 10^9$ km. So, if x is not too close to $nL/2$, we can neglect τ_2 for all terrestrial distances. With this in mind, the formula (3.6) becomes

$$\tau(x) = -\frac{\ell^2}{2\tau_1} \left(1 - \sqrt{1 + \left(\frac{2\tau_1}{\ell} \right)^2} \right), \quad (3.10)$$

while linear approximation yields

$$\tau(x) = \tau_1(x). \quad (3.11)$$

In addition, τ_1 can be rewritten without the term proportional to the extremely small coefficient ω/p_0 . Indeed, the numerical value of this factor in the recent experiments [13, 14, 15] is less than 10^{-22} . Thus, with the big precision, we can write

$$\tau_1 \approx -\frac{\omega\ell^2}{4} \frac{\sin^2 2\theta \sin 2\omega x}{1 - \sin^2 2\theta \sin^2 \omega x}. \quad (3.12)$$

3.2 Graphic illustration

Let us now analyze the consequences of the formula (3.10). As we can see, the neutrino delay $\tau(x)$ takes negative values when $x \in [nL, (n + \frac{1}{2})L]$, positive values when $x \in [(n + \frac{1}{2})L, (n + 1)L]$ and zeros when $x = \frac{n}{2}L$ for all integers $n \geq 0$. For demonstration purposes, let us make a numerical example by choosing our free parameters as follows:

$$p_0 \approx 17 \text{ GeV}, \quad \ell \approx 1.4 \text{ km}. \quad (3.13)$$

With the known values of $\Delta m^2 \approx 2.3 \cdot 10^{-3} \text{ eV}^2$, and $0.92 \lesssim \sin^2 2\theta \lesssim 1$, this yields the behaviour shown in Fig. 1. The negative values of the neutrino delay time τ indicate that

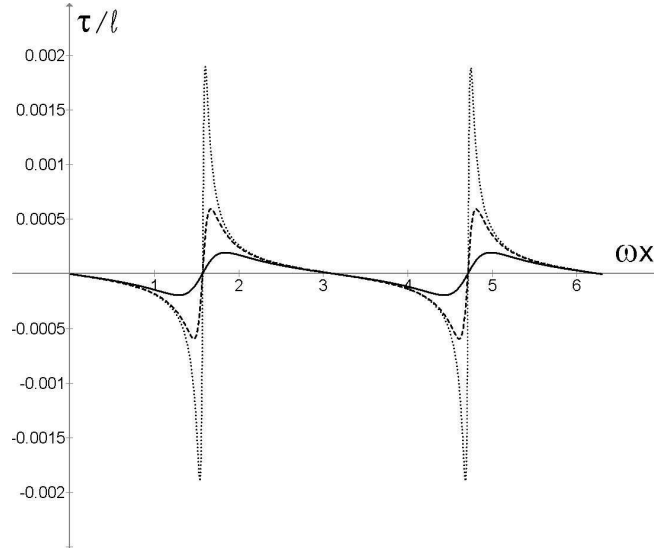


Figure 1: Oscillations of neutrino delay. The three cases are defined by $\sin^2 2\theta = 0.92$ (solid line), $\sin^2 2\theta = 0.99$ (dashed line), and $\sin^2 2\theta = 0.999$ (dotted line).

neutrino may arrive earlier than expected on the basis of the average velocity \bar{v} . Having

in mind that $\bar{v} \approx 1$, we expect that the neutrino velocity

$$v(x) \equiv \frac{dx}{dt} \approx 1 - \frac{d\tau(x)}{dx},$$

may periodically exceed the speed of light. Indeed, differentiating (3.10), we obtain

$$v(x) = 1 + 4 \frac{\sqrt{1 + \left(\frac{2\tau_1}{\ell}\right)^2} - 1}{\sqrt{1 + \left(\frac{2\tau_1}{\ell}\right)^2}} \frac{1 - (1 + \cos^2 2\theta) \sin^2 \omega x}{\sin^2 2\theta \sin^2 2\omega x}.$$

As expected, the neutrino velocity $v(x)$ is a periodic function of x , with the period L . Its behavior is shown in Fig. 2. As we can see, most of the time it exceeds the speed of light.

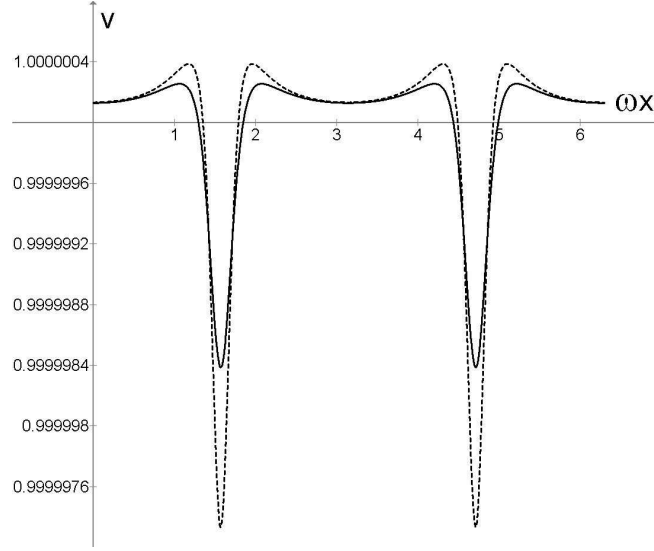


Figure 2: Oscillations of neutrino velocity. The two cases are defined by $\sin^2 2\theta = 0.92$ (solid line), and $\sin^2 2\theta = 0.95$ (dashed line).

In particular, $v(x) > 1$ for all $x \ll L$. We see that averaging $v(x)$ over small distances results in superluminal effective speed. The effective speed v_{eff} is defined as

$$v_{eff} \equiv \frac{x(t)}{t} \approx 1 - \frac{\tau(x)}{x}. \quad (3.14)$$

As opposed to v , that can not be directly measured, the effective speed v_{eff} can. In fact, it is v_{eff} that has been measured in the experiments [13, 14, 15]. Using (3.10) in (3.14), we easily calculate the formula for the effective speed. Its graph is displayed in Fig. 3. As we can see, it is not exactly a periodic function, but nevertheless, it periodically exceeds the speed of light.

Let us now calculate the extreme values of neutrino delays and velocities. To simplify exposition, we shall restrict our analysis to one period $x \in [0, L]$. Then, if the extreme points are denoted by x_{\pm} ($\tau_{min} \equiv \tau(x_-)$, $\tau_{max} \equiv \tau(x_+)$), one finds $x_+ + x_- = L$, and $\tau_{max} = -\tau_{min}$. The extreme points x_+ and x_- are obtained by solving the equation

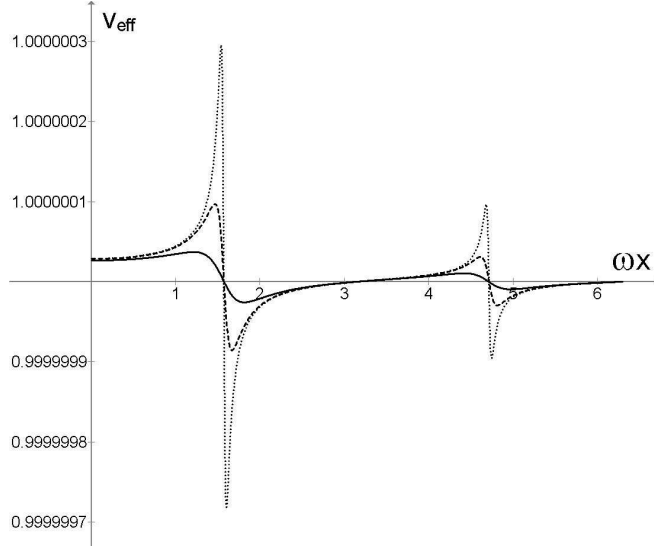


Figure 3: Oscillations of neutrino effective velocity. The three cases are defined by $\sin^2 2\theta = 0.92$ (solid line), $\sin^2 2\theta = 0.99$ (dashed line), and $\sin^2 2\theta = 0.999$ (dotted line).

$d\tau/dx = 0$, whereupon

$$x_{\pm} \approx \frac{L}{2} \left[1 \pm \left(1 - \frac{4\theta}{\pi} \right) \right],$$

and the function $\tau_{min} \equiv \tau(x_-)$, in the linear approximation in τ/ℓ , takes the value

$$\tau_{min} = -\frac{\omega \ell^2}{4} \frac{\sin^2 2\theta}{\cos 2\theta}.$$

The corresponding effective velocity is given by $v_{eff}(x_-) = 1 - \tau_{min}/x_-$. To illustrate this, let us calculate the minimal value that $\tau(x)$ can possibly have for a given θ , and display the corresponding x_- and v_{eff} . Fixing our free parameters as in the example (3.13), we obtain the Table 1. As we can see, the minimal time neutrino needs to arrive at x depends on the mixing angle θ . So does the corresponding effective velocity v_{eff} . The maximal effect is obtained if θ is close to $\pi/4$. Interestingly enough, the corresponding x turns out to be

$\sin^2 2\theta$	x_-	τ_{min}	v_{eff}
0.92	7,400 km	-1 ns	$1 + 1.48 \cdot 10^{-7}$
0.99	8,400 km	-3 ns	$1 + 4.22 \cdot 10^{-7}$
0.999	8,820 km	-10 ns	$1 + 13.5 \cdot 10^{-7}$
0.9999	8,940 km	-32 ns	$1 + 42.2 \cdot 10^{-7}$

Table 1: θ dependence of the minimal neutrino delay

close to the value $x = L/2$ for all the displayed θ . Unfortunately, the probability density P_μ in $x = L/2$ goes to zero when $\theta \rightarrow \pi/4$. Indeed, it is seen from (2.10) that complete disappearance of muon neutrino is possible only if $\theta = \pi/4$. Whenever this happens, a pure tau neutrino appears instead.

Similarly, the maximal value of the velocity $v(x)$ is obtained by solving the equation $dv/dx = 0$. It is shown that v_{max} increases with θ , approaching its absolute maximum $v_{max} = 1 + \frac{2}{3}$ when $\theta \rightarrow \pi/4$. At the same time, $x \rightarrow L/2$. Sadly, the probability P_μ to detect muon neutrino in the point where it reaches its maximal velocity is zero. In other words, *while accelerating towards $v_{max} = 1 + \frac{2}{3}$, the muon neutrino gradually disappears*.

As for the maximum of the effective velocity v_{eff} , it is obtained as the solution of the equation $dv_{eff}/dx = 0$. Again, the maximum of v_{eff} is a function of θ , which approaches its absolute maximum $(v_{eff})_{max} = 1 + 2\ell/L$ in the limit $\theta \rightarrow \pi/4$. At the same time, $x \rightarrow L/2$, and the probability density P_μ goes to zero.

Finally, let us mention that the case $\theta = \pi/4$ is completely different from the case $\theta \rightarrow \pi/4$. Indeed, when $\theta = \pi/4$, the velocity $v(x)$ always exceeds the speed of light. Its behaviour is shown in Fig. 4. As we can see, it reaches its maximal value $v_{max} = 2$ in

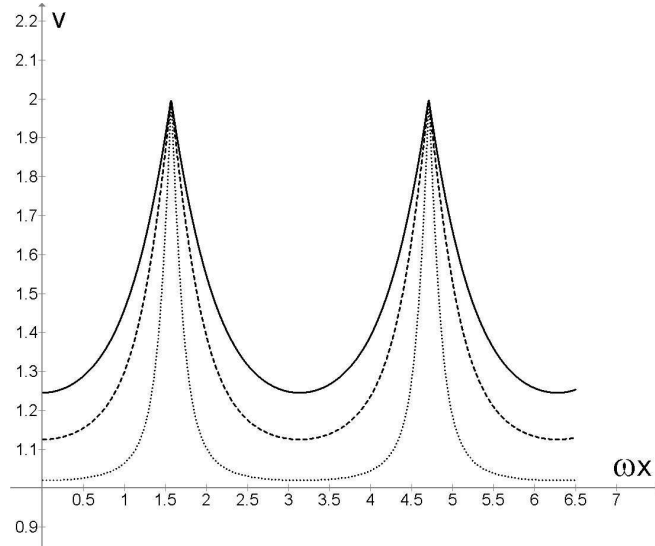


Figure 4: Oscillations of neutrino velocity for $\theta = \pi/4$. The three cases are defined by $\omega\ell = 0.7$ (solid line), $\omega\ell = 0.5$ (dashed line), and $\omega\ell = 0.2$ (dotted line).

$x = L/2$, independently of the neutrino energy. Unfortunately, the probability density to detect $v_{max} = 2$ turns out to be zero. Again, this is because the muon neutrino turns into tau neutrino in $x = L/2$.

4. The shape of the wave packet

Let us analyze how our results depend on the shape of the wave packet. It is useful to introduce some parameter $\gamma \geq 0$, so that variation of γ from zero to infinity changes the shape of the wave packet from almost rectangular to very sharp. Let us generalize (3.4) by choosing

$$\rho(\tau) \propto \begin{cases} \left(1 - \frac{\tau^2}{\ell^2}\right)^\gamma, & -\ell < \tau < \ell \\ 0, & \text{otherwise} . \end{cases} \quad (4.1)$$

To check if the corresponding momentum distribution is sharp around $p = p_0$, as required by our introductory assumptions, we calculate the Fourier transform

$$a(p) = \frac{1}{\sqrt{2\pi}} \int d\tau \rho(\tau) e^{i(p-p_0)\tau}. \quad (4.2)$$

The resulting expression has the form

$$a(p_0 + p) \propto \frac{J_{\gamma+\frac{1}{2}}(\ell p)}{(\ell p)^{\gamma+\frac{1}{2}}}, \quad (4.3)$$

where $J_{\gamma+\frac{1}{2}}(p)$ are ordinary Bessel functions of the order $\gamma + \frac{1}{2}$. Note that $a(p)$ is real and finite function of its argument. For integer values of γ , it can be expressed in terms of elementary functions. For example, the expression for $\gamma = 0$ reads

$$a(p_0 + p) \propto \frac{\sin \ell p}{\ell p}, \quad (4.4)$$

and the expression for $\gamma = 1$ has the form

$$a(p_0 + p) \propto \frac{\sin \ell p}{(\ell p)^3} (1 - \ell p \cot \ell p). \quad (4.5)$$

The necessary and sufficient condition for both amplitudes to be well localized around $p = p_0$ can be expressed in terms of the dimensionless quantity ℓp_0 . It reads

$$\ell p_0 \gg 1. \quad (4.6)$$

Indeed, in the limit $\ell p_0 \rightarrow \infty$, the distributions (4.4) and (4.5) take the form $a(p) \propto \delta(p - p_0)$. As an illustration, the data from the recent experiments [13, 14, 15] yield $\ell p_0 > 10^{14} \gg 1$, telling us that the condition (4.6) is indeed satisfied. In conclusion, the momentum distributions for the whole interval $0 < \gamma < 1$ are sharply localized.

Now that we are convinced that the approximation we have chosen to work with holds true, we proceed to solve the equation (3.1) for the set of parameter dependent amplitudes (4.1). Neglecting terms proportional to the small quantity τ_2 , the equation (3.1) turns into the quadratic equation

$$\tau_1 \tau^2 + \gamma \ell^2 \tau - \tau_1 \ell^2 = 0,$$

which differs from the corresponding equation of the case $\gamma = 1$ by the simple replacement $\tau_1 \rightarrow \tau_1/\gamma$. Having this in mind, we easily obtain

$$\tau(x) = -\frac{\gamma \ell^2}{2\tau_1} \left(1 - \sqrt{1 + \left(\frac{2\tau_1}{\gamma \ell} \right)^2} \right). \quad (4.7)$$

In the lowest approximation in τ/ℓ , we come to

$$\tau(x) = \frac{\tau_1}{\gamma}. \quad (4.8)$$

The requirement that (4.8) has a dominant role in (4.7) puts a restriction on γ . For the experiments [13, 14, 15], a sufficient condition to ensure the validity of (4.8) is

$$\gamma \gg 10^{-8}.$$

Finally, let us say something about how neutrino velocity depends on γ . We have seen in the preceding section that v_{max} is a discontinuous function in $\theta = \pi/4$. The same happens in the case $\gamma \neq 1$. Indeed, the neutrino velocity has two absolute maximums,

$$\begin{aligned} v_{max} &= 1 + \frac{2\gamma}{3} & \text{when } \theta \rightarrow \frac{\pi}{4}, \\ v_{max} &= 1 + \gamma & \text{when } \theta = \frac{\pi}{4}, \end{aligned}$$

which both take place in $x = L/2$. In this point, however, the probability density P_μ goes to zero. Thus, we can say that *muon neutrino, which approaches the point of absolutely maximal velocity, gradually disappears*. This is because it turns into a tau neutrino.

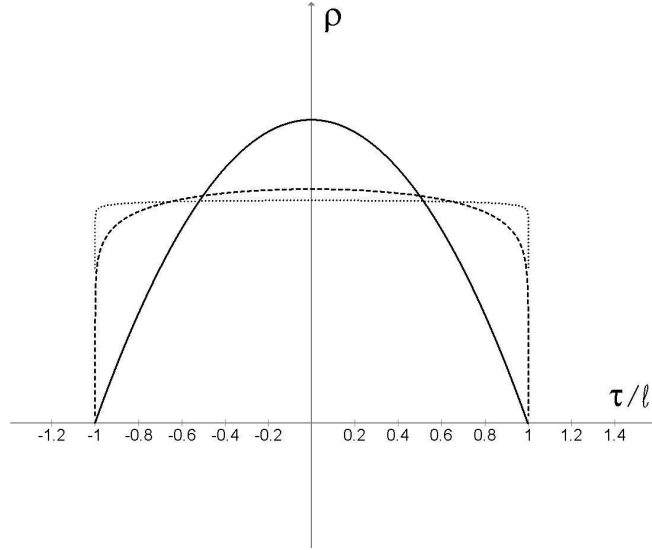


Figure 5: Shapes of the neutrino wave packet. The three cases are defined by $\gamma = 1$ (solid line), $\gamma = 0.1$ (dashed line), and $\gamma = 0.01$ (dotted line).

In conclusion, as γ approaches zero, the wave packet is more sharply localized in momentum space, and has more rectangular shape in coordinate space. (See the example in Fig. 5.) This makes the neutrino delay time $|\tau|$ increase. As a consequence, the superluminal effect becomes easier to detect. Still, as the known experiments lack the parameter that determines the shape of the neutrino wave packet, the comparison with measurements remains an uneasy problem.

At the end, let us note that the above results refer to the detection of muon neutrino. The generalization to the case of tau neutrinos is straightforward. In the next section, we shall analyze the evolution of tau component of initially pure muon neutrino.

5. Velocity oscillations of tau neutrinos

In the preceding section, we have considered the dynamics of initially pure muon neutrino, and calculated the probability to detect muon neutrino at later times. In a similar way, we can derive the probability density to detect tau neutrino. It is defined by

$$P_\tau(x, t) = |\langle \nu_\tau(x) | \nu_\mu(t) \rangle|^2 ,$$

where $|\nu_\tau(x)\rangle \equiv \sin \theta |1\rangle |x\rangle + \cos \theta |2\rangle |x\rangle$ is the eigenstate of the position operator for tau neutrino. The direct calculation yields

$$P_\tau(x, t) = \rho^2(\bar{v}\tau) \sin^2 2\theta \sin^2 \omega t . \quad (5.1)$$

As we can see, no correction linear in Δv appears in (5.1). It is checked that

$$\int (P_\mu + P_\tau) dx = \int \rho^2(x) dx = 1 ,$$

as it should be in the approximation we work with.

If the neutrino detector detects tau neutrinos instead of muon neutrinos, the corresponding equation $\partial P_\tau / \partial t = 0$ takes the form

$$\frac{\partial_\tau \rho^2}{\rho^2} + \frac{4}{\ell^2} \tau_3(x) = 0 , \quad (5.2)$$

where

$$\tau_3(x) \equiv \frac{\omega \ell^2}{4} \cot \omega x . \quad (5.3)$$

As we can see, the dependence on the mixing angle θ does not appear in (5.3). Choosing the same form for ρ as in (4.1), we obtain the expression of the same form as (4.7), with the only difference that $\tau_3(x)$ is used instead of $\tau_1(x)$. The approximation linear in τ/ℓ then yields

$$\tau_\tau(x) = \frac{\tau_3(x)}{\gamma} . \quad (5.4)$$

If we restrict our analysis to distances which are not too close to $(n + \frac{1}{2})L$, and the mixing angle θ is close to $\pi/4$, the corresponding formula for τ_1 takes the simplified form $\tau_1 \approx -\frac{\omega \ell^2}{2} \tan \omega x$. It then yields

$$\tau_1(x) \tau_3(x) \approx -\frac{\omega^2 \ell^4}{8} = \text{const.} , \quad (5.5)$$

or equivalently,

$$\tau_\mu(x) \tau_\tau(x) \approx -\frac{\omega^2}{8\gamma^2} [\ell^2 - \tau_\mu^2(x)] [\ell^2 - \tau_\tau^2(x)] . \quad (5.6)$$

As we can see, the arrival times τ_μ and τ_τ for muon and tau neutrinos have opposite signs. Thus, when one of them has superluminal speed, the speed of the other is subluminal. The relation (5.6) nicely illustrates connection between oscillations of muon and tau neutrinos. When one flavor has minimal deviation ($\tau_\mu(x) \rightarrow 0$) the other has maximal one ($\tau_\tau(x) \rightarrow \ell$).

(Note, however, that our results are derived under the assumption that τ is not too close either to ℓ or to zero.)

Finally, let us point out another interesting relation between $\tau_1(x)$ and $\tau_3(x)$:

$$\tau_1(x + L/2) \approx 2\tau_3(x).$$

With this, the calculations related to tau neutrino are greatly simplified. In particular, the linear approximation in τ/ℓ yields the relation $\tau_\mu(x + L/2) = 2\tau_\tau(x)$, showing that the double delay time of tau neutrino is obtained by the simple shifting $x \rightarrow x + L/2$ in the expression for the delay time of muon neutrino.

6. Summary and discussion

In this paper, we have demonstrated that the flight of a free neutrino is characterized not only by the well known flavor oscillations, but also by the oscillations of neutrino velocity. This has been done by considering the free evolution of initially pure muon neutrino. First, we calculated the probability density for detecting muon neutrino by the detector placed in a fixed position x . Such a probability distribution was a function of time, and its maximum was identified with the moment neutrino arrived at x . This way, we obtained the formula for the evaluation of time the neutrino needed to fly across the distance x .

It should be noted that our formula differs from similar formulae found in literature (see, for example, [10, 11, 12]). The reason for this is the difference in our definitions of the neutrino position. While we identify it with the maximum of its probability distribution, the authors of [10, 11, 12] use the average of neutrino position operator. As a consequence, our formula carries additional dependence on the size and shape of the neutrino wave packet. The neutrino delay, as compared to the arrival time of the photon, was found to be a periodic function of x , with the period which did not depend on the size and shape of the neutrino wave packet. The neutrino velocity, on the other hand, turned out to drastically depend on the size and shape of the wave packet: the more rectangular the shape of the packet was, the higher velocity neutrino could reach. As it turned out, the *neutrino velocity periodically exceeded the speed of light*. We were able to derive the maximal velocity the neutrino could possibly achieve during its flight. As it turned out, the probability to detect maximal velocity was rather small, approaching zero when $\theta \rightarrow \pi/4$.

The fact that our formula depends on the size and shape of the neutrino wave packet makes the comparison with the experiment very difficult. This is because the known experiments do not provide the information on these two parameters. Nevertheless, we shall try to test our formula by comparing its predictions with three recent experiments [13, 14, 15]. To this end, we shall estimate the size of the neutrino wave packet using the results found in literature. In particular, a useful study of the properties of accelerator neutrinos can be found in Ref. [16]. Using the size of the wave packet as found in [16], and almost Gaussian shape as defined by $\gamma = 1$, we make the Table 2. As we can see, our theoretical predictions do not contradict any of the three experiments. Even if the neutrino wave packet is taken 10^5 times longer, the agreement with the experiment is not compromised. The same holds when it comes to the wave packet shape. Indeed,

Experiment	p_0	2ℓ	x	$v_{exp} - 1$	$v_{eff} - 1$
MINOS	3 GeV	7.7 cm	734 km	$(5.1 \pm 7.5) \cdot 10^{-5}$	$0.9 \cdot 10^{-15}$
ICARUS	17 GeV	0.67 cm	730 km	$(0.1 \pm 5.7) \cdot 10^{-6}$	$1.5 \cdot 10^{-19}$
OPERA	17 GeV	0.67 cm	730 km	$(2.7 \pm 6.5) \cdot 10^{-6}$	$1.5 \cdot 10^{-19}$

Table 2: Comparison with experiments

the change of γ in the allowed interval $\gamma \gg 10^{-8}$ leaves the theoretical values within the experimental error. In fact, it is the big experimental error that basically ensures this agreement. For a real test of our theoretical predictions, more efficient measurements are needed. In particular, our equations suggest how neutrino free parameters should be chosen to maximize the superluminal effect.

Let us say something about energy dependence of the neutrino velocity. As seen from our formulae, the energy dependence of $v_{eff} - 1$ has oscillating character. To simplify the analysis, we shall restrict to the region $x \ll L$, which is achieved by using short range high-energy neutrinos. Upon this, the measured quantity $v_{eff} - 1$ becomes proportional to ℓ^2/p_0^2 , which reduces to ℓ^2/E_0^2 in the ultrarelativistic limit. At the same time, the wave packet size ℓ is also energy dependent. Indeed, it has been shown in [16] that, depending on the experimental details, ℓ is proportional to either $1/E_0$ or $1/E_0^2$. Thus, in the ultrarelativistic limit,

$$v_{eff} - 1 \sim \frac{1}{E_0^4} \quad \text{or} \quad v_{eff} - 1 \sim \frac{1}{E_0^6}.$$

This is a strong energy dependence, but still undetectable by the ICARUS and OPERA experiments. (MINOS experiment is excluded from the analysis because it violates the requirement $x \ll L$.) Indeed, it is seen from Table 2 that the values of $v_{eff} - 1$ are 10^{13} times smaller than $v_{exp} - 1$, which implies that *100 times lower energy still yields the result that agrees with the experiments*. Higher energies, on the other hand, diminish the value of $v_{eff} - 1$, and therefore, fit the experiments even better.

Finally, let us comment on the result of Ref. [17] stating that superluminal neutrinos should rapidly loose energy during their flight. This result has been derived with the assumption that neutrino speed does not change along neutrino trajectory. However, we have shown that this is not the case. In general, the neutrino velocity has oscillating character, but we shall simplify the analysis by adopting the condition $x \ll L$. In this regime, the results of the preceding sections yield

$$\delta \approx 2\alpha \frac{d}{dx} \left(\frac{x}{E^2} \right), \quad \alpha \equiv \frac{1}{2\gamma} \left(\frac{\Delta m^2 \ell}{4} \right)^2,$$

where $\delta \equiv v^2 - 1$. Using this expression in the formula for the rate at which superluminal neutrinos loose their energy [17],

$$\frac{dE}{dx} = -\kappa E^6 \delta^3, \quad \kappa \equiv \frac{25}{448} \frac{G_F^2}{192\pi^3},$$

we obtain a complicated, higher order differential equation. To simplify the calculations, we shall assume that superluminal neutrinos loose their energy slowly ($|dE/dx| \ll E/x$).

Then, we obtain a simple approximative solution

$$E \approx E_0 - E_T, \quad E_T \equiv 8\kappa\alpha^3 x,$$

telling us that the rate superluminal neutrinos loose their energy is linear in x . Notice, however, that the value of the constant $\kappa\alpha^3$ is extremely small in the present experiments. For the numerical illustration, we shall take data from Table 2. With the known value of the Fermi coupling constant $G_F \approx 1.17 \cdot 10^{-5} \text{ GeV}^{-2}$, and using the almost Gaussian wave packet as defined by $\gamma = 1$, we find

$$E_T \lesssim 1.1 \cdot 10^{-35} \text{ GeV}.$$

Having in mind that the initial neutrino energy in the considered experiments [13, 14, 15] is $E_0 \geq 3 \text{ GeV}$, we see that the loss of energy during the flight of superluminal neutrinos is negligible. Our initial assumption $|dE/dx| \ll E/x$ is thereby justified, and the validity of our conclusion is confirmed. We can still change the wave packet shape, but this can not significantly modify our conclusions. Indeed, the lowest allowed value of γ is of the order 10^{-7} , which leads to the value $E_T \sim 10^{-14} \text{ GeV}$. This is still negligible with respect to the initial neutrino energy ($E_T \ll E_0$) in all terrestrial experiments. Summarized, *we have proven the existence of energy conserving superluminal free neutrinos.*

In conclusion, if neutrinos have different masses, the oscillations of neutrino velocity necessarily exist. In particular, the neutrino velocity periodically exceeds the speed of light. The significance of this result is threefold. First, it shows that superluminal speed can be achieved without violation of special relativity. Second, our equations suggest how neutrino parameters should be chosen to maximize the superluminal effect in new experiments. Finally, our formula offers an independent way to determine neutrino mixing angles.

Acknowledgments

This work is supported in part by the Serbian Ministry of Education and Science, under Contract No. 171031.

References

- [1] C. Giunti and C. W. Kim, *Fundamentals of Neutrino Physics and Astrophysics* (Oxford: Oxford Univ. Press, 2007).
- [2] J. J. Evans, *The current status of neutrino mixing*, [arXiv:1107.3846](#)
- [3] S. M. Bilenky and B. Pontecorvo, *The quark-lepton analogy and the muonic charge*, *Sov. J. Nucl. Phys.* **24** (1976) 316.
- [4] S. M. Bilenky and B. Pontecorvo, *Again on neutrino oscillations*, *Lett. Nuovo Cim.* **17** (1976) 569.
- [5] S. Eliezer and A. R. Swift, *Experimental consequences of electron neutrino - muon neutrino mixing in neutrino beams*, *Nucl. Phys.* **B 105** (1976) 45.

- [6] H. Fritzsch and P. Minkowski, *Vector-like weak currents, massive neutrinos, and neutrino beam oscillations*, *Phys. Lett.* **B 62** (1976) 72.
- [7] S. P. Mikheev and A. Yu. Smirnov, *Resonant amplification of oscillations in matter and spectroscopy of solar neutrinos*, *Sov. J. Nucl. Phys.* **42** (1985) 913.
- [8] S. P. Mikheev and A. Yu. Smirnov, *Resonant amplification of neutrino oscillations in matter and solar neutrino spectroscopy*, *Nuovo Cim.* **C9** (1986) 17.
- [9] L. Wolfenstein, *Neutrino oscillations in matter*, *Phys. Rev.* **D 17** (1978) 2369.
- [10] A. Meozi and M. Billini, *Superluminal group velocity of neutrinos*, [arXiv:1110.1253](#).
- [11] T. R. Morris, *Superluminal velocity through near-maximal neutrino oscillations or by being off shell*, *J. Phys.* **G 39** (2012) 045010.
- [12] M. V. Berry, N. Brunner, S. Popescu and P. Shukla, *Can apparent superluminal neutrino speed be explained as a quantum weak measurement?*, *J. Phys.* **A 44** (2011) 492001.
- [13] P. Adamson *et al.* [MINOS Collaboration], *Measurement of neutrino velocity with the MINOS detectors and NuMI neutrino beam*, *Phys. Rev.* **D 76** (2007) 072005.
- [14] M. Antonello *et al.* [ICARUS Collaboration], *Measurement of the neutrino velocity with the ICARUS detector at the CNGS beam*, *Phys. Lett.* **B 713** (2012) 17.
- [15] T. Adam *et al.* [OPERA Collaboration], *Measurement of the neutrino velocity with the OPERA detector in the CNGS beam*, [arXiv:1109.4897v4](#).
- [16] H. Minakata and A. Yu. Smirnov, *Neutrino Velocity and Neutrino Oscillations*, *Phys. Rev.* **D 85** (2012) 113006.
- [17] A. G. Cohen and S. L. Glashow, *New Constraints on Neutrino Velocity*, *Phys. Rev. Lett.* **107** (2011) 181803.